

Detailed Derivation of Fanno Flow Relationships

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Version 1.1

1 October 2013

v1.1 – fixed a sign error on Eq. (5)

Motivation

My motivation for writing this note results from preparing course notes for the SUNY University at Buffalo MAE 422 class – Gasdynamics. The course textbook, *Modern Compressible Flow* by John Anderson, 2ND Edition, is a generally comprehensive and factual textbook, but I found the treatment of the derivation of one-dimensional flow with friction (“Fanno” flow), while giving all correct results, to have skipped over a number of steps in arriving at the final derivations, the path of which was not obvious to me without significant effort. In particular, the derivations of Eqns (3.96) and (3.97) in Section 3.9 were non-trivial (at least to me). This note is an effort to archive my understanding of the derivation complete with all intermediate steps in the hopes that it might help others.

Overview

Fanno flow, or one-dimensional flow with wall friction, considers the equivalent, inviscid effect of a friction force along the walls of a one-dimensional duct by introducing a deficit term into the inviscid momentum equation. The flow is considered to be steady, isentropic (adiabatic plus reversible), without body forces, and shockless. The continuity and momentum equations are identical in form to those used in the derivation of normal shock relations, given in Eqns (1) and (2):

$$\rho_1 u_1 = \rho_2 u_2 \quad (1)$$

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2 \quad (2)$$

Only the momentum equation requires special treatment. The integral form of the momentum equation is given in Eqn (3).

$$\frac{\partial}{\partial t} \iiint_V \rho \vec{V} dV + \iint_S \rho \vec{V} (\vec{V} \cdot d\vec{S}) + \iint_S P d\vec{S} - \vec{F}_{VISCIOUS} = 0 \quad (3)$$

To evaluate Eqn (3) for our specific application, consider the control volume shown in Fig 1. Equation (3) sums the rate of change of momentum within the control volume, the rate at which momentum is carried across the control volume boundary, and the forces acting on the control volume. The pressure force acting on the left and right faces of the control volume is familiar from the derivation of normal shock equations.

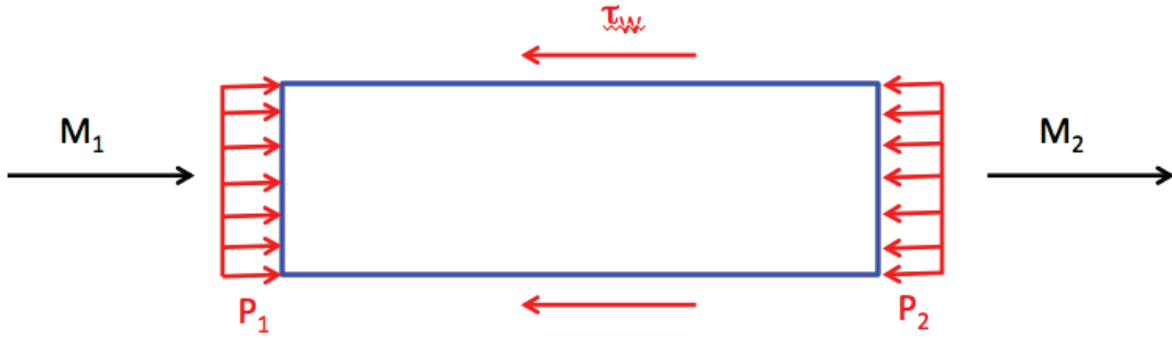


Figure 1. Control Volume for Fanno Flow

We may express each term of Eqn (3), noting that the flow is steady and the first term may be neglected, as written in Eqn (4):

$$\begin{aligned} & \left[\rho_1 (u_1 \hat{i}) (u_1 \hat{i} \cdot -A \hat{i}) \right]_{FACE1} + \left[P_1 (-A \hat{i}) \right]_{FACE1} \\ & + \left[\rho_2 (u_2 \hat{i}) (u_2 \hat{i} \cdot +A \hat{i}) \right]_{FACE2} + \left[P_2 (+A \hat{i}) \right]_{FACE2} - \vec{F}_{VISCIOUS} = 0 \end{aligned} \quad (4)$$

Here, we have left the viscous shear stress term alone momentarily. The vector notation has been preserved in Eqn (4) to carefully preserve the sign conventions. Eliminating the vector notation symbols and cleaning up yields:

$$-\rho_1 u_1^2 A - P_1 A + \rho_2 u_2^2 A + P_2 A - \vec{F}_{VISCIOUS} = 0 \quad (5)$$

The orientation of the viscous friction term may be verified by considering that, for a velocity moving to the right in Fig 1, the friction term will act to the left. By noting that this action of the force shares direction with the pressure force acting on Face 2 of the control volume. Thus, the net friction force term may be represented in Eqn (6):

$$-\rho_1 u_1^2 A - P_1 A + \rho_2 u_2^2 A + P_2 A + \left[\int_0^L C \tau_w dx \right] = 0 \quad (6)$$

Here, a shear stress (force per unit surface area) is integrated across a duct with a circumference, C and a length, L. At this point, we have not specified the shape of the duct. Moving the terms related to Face 1 to the right-hand side yields the momentum equation for Fanno flow, Eqn (7).

$$\boxed{\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 + \frac{C}{A} \int_0^L \tau_w dx} \quad (7)$$

Differential Form of the Momentum Equation

Anderson and many other authors define the friction term specifically for a cylindrical duct. This specification is not strictly necessarily, but we may adhere to the common form by noting that, for a cylindrical duct of diameter D, circumference $C=\pi D$ and area $A=\pi D^2/4$. We may

substitute to obtain the common form while maintaining generality by noting that this implies that $C/A = 4/D$:

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 + \frac{4}{D} \int_0^L \tau_w dx \quad (8)$$

A key to understanding Fanno flow is that the shear stress is a function of velocity and density of the gas, so, as the state of the flow changes under the action of the shear force, the magnitude of the shear force itself changes. Therefore, the momentum equation, Eqn (8) must be placed in differential form before it can be modified further. To do so, it is convenient to group static pressure and dynamic pressure terms together, e.g.:

$$(P_2 - P_1) + (\rho_2 u_2^2 - \rho_1 u_1^2) = -\frac{4}{D} \int_0^L \tau_w dx \quad (9)$$

The differential form of Eqn (9) can be obtained by taking the limit as $L \rightarrow 0$.

$$dP + d(\rho u^2) = -\frac{4}{D} \tau_w dx \quad (10)$$

You can verify this by integrating Eqn (10) between states 1 and 2 (0 to L) to obtain Eqn (9). Although we know that the value of shear stress will be changing with position in the duct, we assume that the **shear stress coefficient**, f , is constant, which is the ratio of shear stress to local dynamic pressure. Thus, Eqn (10) may be cast in terms of f instead:

$$dP + d(\rho u^2) = -\frac{4}{D} \left(\frac{1}{2} \rho u^2 f \right) dx \quad (11)$$

We also know from the continuity equation, Eqn (1), that the product of density times velocity is constant, so it may be extracted from the differential operator:

$$\boxed{dP + \rho u du = -\frac{1}{2} \rho u^2 \left[\frac{4}{D} f dx \right]} \quad (12)$$

Intermediate Thermodynamic Relationships

To operate on Eqn (12) further, we need to obtain a number of intermediate relations that are skipped over in Anderson's book. We will derive each very carefully here.

Energy Equation

Recall that the energy equation in Eqn (3) specifies that total enthalpy is a constant in the flow. It can be expressed in terms of temperature:

$$C_p T + \frac{1}{2} u^2 = h_0 \quad (13)$$

Taking the differential of all terms yields:

$$C_p dT + \frac{1}{2}d(u^2) = 0 \quad (14)$$

Note that we leave the differential in terms of u^2 rather than u . Substituting for the specific heat yields:

$$\frac{\gamma R}{\gamma - 1} dT + \frac{1}{2}d(u^2) = 0 \quad (15)$$

Introducing temperature in the first term yields:

$$\frac{\gamma R T}{\gamma - 1} \frac{dT}{T} + \frac{1}{2}d(u^2) = 0 \quad (16)$$

The speed of sound may be substituted into the first term, yielding:

$$\frac{a^2}{\gamma - 1} \frac{dT}{T} + \frac{1}{2}d(u^2) = 0 \quad (17)$$

The speed of sound may be moved into the velocity differential term, yielding:

$$\frac{dT}{T} + \frac{\gamma - 1}{2} \frac{1}{a^2} d(u^2) = 0 \quad (18)$$

Finally, the Mach number may be introduced, yielding:

$$\boxed{\frac{dT}{T} + \frac{\gamma - 1}{2} M^2 \frac{d(u^2)}{u^2} = 0} \quad (19)$$

Mach Number

The definition of Mach number is used to obtain the next required differential. Start from Eqn (20).

$$a^2 M^2 = u^2 \quad (20)$$

The speed of sound may be defined, yielding:

$$\gamma R T M^2 = u^2 \quad (21)$$

The differential form of Eqn (21) yields:

$$\gamma R M^2 dT + \gamma R T d(M^2) = d(u^2) \quad (22)$$

Temperature is introduced into the first term, yielding:

$$\gamma R T M^2 \frac{dT}{T} + \gamma R T d(M^2) = d(u^2) \quad (23)$$

The speed of sound may be substituted back into the equation, yielding:

$$a^2 M^2 \frac{dT}{T} + a^2 d(M^2) = d(u^2) \quad (24)$$

Finally, dividing by $a^2 M^2$ and noting that $a^2 M^2$ is equal to u^2 yields:

$$\boxed{\frac{dT}{T} + \frac{d(M^2)}{M^2} = \frac{d(u^2)}{u^2}} \quad (25)$$

Continuity Equation

The continuity equation says that the product of density times velocity equals a constant (arbitrarily given as K here).

$$\rho u = K \quad (26)$$

The differential form of the continuity equation yields:

$$u d\rho + \rho du = 0 \quad (27)$$

The equation may be divided by the constant ρu :

$$\frac{d\rho}{\rho} + \frac{du}{u} = 0 \quad (28)$$

We earlier used the differential of u^2 , so we will transform this equation by modifying the 2nd term as follows:

$$\frac{d\rho}{\rho} + \frac{2u du}{2u u} = 0 \quad (29)$$

Finally, we may note that $2 u du$ is equal to $d(u^2)$:

$$\boxed{\frac{d\rho}{\rho} + \frac{d(u^2)}{2u^2} = 0} \quad (30)$$

Ideal Gas Law

The ideal gas law for a thermally perfect gas is:

$$P = \rho RT \quad (31)$$

The differential form of this equation is:

$$dP = \rho R dT + R T d\rho \quad (32)$$

Equation (32) can be divided by pressure, and, using the ideal gas law, yields:

$$\boxed{\frac{dP}{P} = \frac{dT}{T} + \frac{d\rho}{\rho}} \quad (33)$$

Differential Momentum Equation in Terms of Mach Number

The differential relationships, Eqns (19), (25), (30), and (33) can be used to transform the differential momentum equation Eqn (12) into a function of only Mach number. This is accomplished as follows –

Substituting Eqn (33) into Eqn (19) yields:

$$\left[\frac{dP}{P} - \frac{d\rho}{\rho} \right] + \frac{\gamma-1}{2} M^2 \frac{d(u^2)}{u^2} = 0 \quad (34)$$

Substituting Eqn (30) into Eqn (34) yields:

$$\frac{dP}{P} - \left[-\frac{d(u^2)}{2u^2} \right] + \frac{\gamma-1}{2} M^2 \frac{d(u^2)}{u^2} = 0 \quad (35)$$

Combining terms in Eqn (35) yields:

$$\frac{dP}{P} + \left[\frac{1+(\gamma-1)M^2}{2} \right] \frac{d(u^2)}{u^2} = 0 \quad (36)$$

Now, we must deal with the form of the momentum equation that we last found in Eqn (12). This equation can be divided by dynamic pressure, yielding:

$$\frac{dP}{\frac{1}{2}\rho u^2} + \frac{\rho u du}{\frac{1}{2}\rho u^2} = -\frac{4}{D} f dx \quad (37)$$

We know from compressible flow that the dynamic pressure, $\frac{1}{2}\rho u^2$, is equal to $\frac{\gamma}{2}PM^2$. With this substitution, Eqn (37) becomes:

$$\frac{dP}{\frac{\gamma}{2}PM^2} + \frac{2u du}{u^2} = -\frac{4}{D} f dx \quad (38)$$

Just as with the derivation of Eqn (30), we may place the differential in terms of u^2 rather than u .

$$\frac{2}{\gamma M^2} \frac{dP}{P} + \frac{d(u^2)}{u^2} = -\frac{4}{D} f dx \quad (39)$$

Combining Eqns (36) and (39) removes the differential pressure term.

$$\frac{2}{\gamma M^2} \left\{ - \left[\frac{1 + (\gamma - 1) M^2}{2} \right] \frac{d(u^2)}{u^2} \right\} + \frac{d(u^2)}{u^2} = - \frac{4}{D} f dx \quad (40)$$

Combining terms yields:

$$\left\{ 1 - \left[\frac{1 + (\gamma - 1) M^2}{\gamma M^2} \right] \right\} \frac{d(u^2)}{u^2} = - \frac{4}{D} f dx \quad (41)$$

Using a common denominator of γM^2 allows partial cancellation of the numerator within the brackets.

$$\left\{ \frac{\gamma M^2 - 1 - \gamma M^2 + M^2}{\gamma M^2} \right\} \frac{d(u^2)}{u^2} = - \frac{4}{D} f dx \quad (42)$$

or

$$\left\{ \frac{M^2 - 1}{\gamma M^2} \right\} \frac{d(u^2)}{u^2} = - \frac{4}{D} f dx \quad (43)$$

Finally, Eqns (19) and (25) may be combined to yield:

$$\left[\frac{d(u^2)}{u^2} - \frac{d(M^2)}{M^2} \right] + \frac{(\gamma - 1)}{2} M^2 \frac{d(u^2)}{u^2} = 0 \quad (44)$$

Solving Eqn (44) for the $d(M^2)$ term yields:

$$\frac{d(M^2)}{M^2} = \left[1 + \frac{(\gamma - 1)}{2} M^2 \right] \frac{d(u^2)}{u^2} \quad (45)$$

Equation (45) can be substituted into Eqn (43) to yield the final equation in terms of Mach number only,

$$\left\{ \frac{M^2 - 1}{\gamma M^2} \right\} \left\{ \left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^{-1} \frac{d(M^2)}{M^2} \right\} = - \frac{4}{D} f dx \quad (46)$$

where we note that the term preceding the $d(M^2)$ term in brackets has a -1 exponent to imply division. The Anderson book expresses this relationship in terms of dM rather than $d(M^2)$, which we may obtain by employing the product rule on that term.

$$\frac{(M^2 - 1)}{\gamma M^2} \left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^{-1} \frac{2 M dM}{M^2} = - \frac{4}{D} f dx \quad (47)$$

Finally, we may transfer the minus sign to the left side and clean up the equation to yield the exact form given by Anderson as 3.96.

$$\boxed{\frac{2(1-M^2)}{\gamma M^2} \left[1 + \frac{(\gamma-1)}{2} M^2 \right]^{-1} \frac{dM}{M} = \frac{4}{D} f dx} \quad (48)$$

Integration of Momentum Equation, Eqn (48)

The integral of the right hand side of Eqn (48) is trivial, so we will concern ourselves with the integral of the left hand side of Eqn (48). It is actually slightly easier to work with the integral in terms of M^2 instead of M , so we will concern ourselves with an integral of the form given in Eqn (49):

$$\int_1^2 \frac{(1-M^2)}{\gamma M^2} \left[1 + \frac{(\gamma-1)}{2} M^2 \right]^{-1} \frac{d(M^2)}{M^2} = \int_1^2 \frac{4}{D} f dx \quad (49)$$

Defining an intermediate variable, z , as M^2 simplifies the notation slightly:

$$\int_1^2 \frac{(1-z)}{\gamma z^2 \left(1 + \frac{(\gamma-1)}{2} z \right)} dz \quad (50)$$

We add and subtract some terms from the numerator to parse the integral:

$$\int_1^2 \frac{(1-z) + \left(1 + \frac{(\gamma-1)}{2} z \right) - \left(1 + \frac{(\gamma-1)}{2} z \right)}{\gamma z^2 \left(1 + \frac{(\gamma-1)}{2} z \right)} dz \quad (51)$$

The numerator can be rearranged to cancel some of the terms:

$$\int_1^2 \frac{\left(1 + \frac{(\gamma-1)}{2} z \right) + \left(1 - z - 1 - \frac{\gamma}{2} z + \frac{1}{2} z \right)}{\gamma z^2 \left(1 + \frac{(\gamma-1)}{2} z \right)} dz \quad (52)$$

By placing the integrand in this form, the integral can be split into two sections:

$$\int_1^2 \frac{\left(1 + \frac{(\gamma-1)z}{2}\right)}{\gamma z^2 \left(1 + \frac{(\gamma-1)z}{2}\right)} dz + \int_1^2 \frac{-\frac{1}{2}(\gamma+1)z}{\gamma z^2 \left(1 + \frac{(\gamma-1)z}{2}\right)} dz \quad (53)$$

Cancelling the common term in the first integral yields:

$$\int_1^2 \frac{1}{\gamma z^2} dz - \frac{(\gamma+1)}{2\gamma} \int_1^2 \frac{1}{z \left(1 + \frac{(\gamma-1)z}{2}\right)} dz \quad (54)$$

We can now perform an integration substitution on the second part by defining yet another new variable, y , in Eqn (55):

$$y \equiv \frac{z}{\left(1 + \frac{(\gamma-1)z}{2}\right)} \quad (55)$$

It's derivative equals (using the quotient rule):

$$\frac{dy}{dz} \equiv \frac{\left(1 + \frac{(\gamma-1)z}{2}\right)(1) - z \frac{(\gamma-1)}{2}}{\left(1 + \frac{(\gamma-1)z}{2}\right)^2} = \frac{1}{\left(1 + \frac{(\gamma-1)z}{2}\right)^2} \quad (56)$$

By multiplying and dividing by z , we obtain:

$$dy = \frac{z}{\left(1 + \frac{(\gamma-1)z}{2}\right)} \frac{dz}{z \left(1 + \frac{(\gamma-1)z}{2}\right)} \quad (57)$$

The original definition of y may be substituted into Eqn (57), yielding Eqn (58):

$$\frac{dy}{y} = \frac{dz}{z \left(1 + \frac{(\gamma-1)z}{2}\right)} \quad (58)$$

Making this substitution into the integral equation, Eqn (54), yields:

$$\int_1^2 \frac{1}{\gamma z^2} dz - \frac{(\gamma+1)}{2\gamma} \int_1^2 \frac{dy}{y} \quad (59)$$

Placed in this form, the equation may be readily integrated to yield Eqn (60).

$$-\frac{1}{\gamma z} \Big|_1^2 - \frac{(\gamma+1)}{2\gamma} \ln|y| \Big|_1^2 \quad (60)$$

Finally, we may use the definitions of z and y and substitute the result back into our original target equation to obtain Eqn (61):

$$-\frac{1}{\gamma M^2} \Big|_1^2 - \frac{(\gamma+1)}{2\gamma} \ln \left| \frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right| \Big|_1^2 \quad (61)$$

Equation (61) is used to evaluate the integral given in Eqn (49):

$$\left[-\frac{1}{\gamma M^2} - \frac{(\gamma+1)}{2\gamma} \ln \left| \frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right| \right] \Big|_1^2 = \frac{4}{D} f(x_2 - x_1) \quad (62)$$

Equation (62) represents the fully-integrated momentum equation for Fanno flow.

Choked Flow Duct Length

Fanno flow is typically analyzed by computing the dimensionless length required to choke the flow (the point where Mach number becomes 1.0), denoted by the superscript “*”. This length, L^* , may be found by using Eqn (62) and evaluating point 1 at any arbitrary Mach number, M , to point 2, where $M^* = 1$. Doing so yields Eqn (63):

$$\frac{4}{D} f(L^* - 0) = \left[-\frac{1}{\gamma(1)} - \frac{(\gamma+1)}{2\gamma} \ln \left(\frac{1}{1 + \frac{\gamma-1}{2}(1)} \right) \right] - \left[-\frac{1}{\gamma M^2} - \frac{(\gamma+1)}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \right] \quad (63)$$

Note also that we have dropped the absolute value on the natural log since we can recognize from Eqn (62) that, for any value of M , the argument will be positive. The natural log terms may be grouped together, yielding Eqn (64):

$$\frac{4}{D} fL^* = \frac{1}{\gamma M^2} - \frac{1}{\gamma} + \frac{(\gamma+1)}{2\gamma} \left\{ \ln \left(\frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) - \ln \left(\frac{1}{\frac{2}{2} + \frac{\gamma-1}{2}} \right) \right\} \quad (64)$$

By noting the the difference between two natural logs is equal to the ratio of the two arguments {e.g. $\ln a - \ln b = \ln (a/b)$ }, we can simplify the expression to:

$$\frac{4}{D} fL^* = \frac{1}{\gamma M^2} - \frac{1}{\gamma} + \frac{(\gamma+1)}{2\gamma} \ln \left(\frac{(\gamma+1)}{2} \frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \right) \quad (65)$$

With minor cleanup, Eqn (65) becomes:

$$\boxed{\frac{4fL^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{(\gamma+1)}{2\gamma} \ln \left(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \right)} \quad (66)$$

Property Ratios

Finally, we may express ratios of any two properties between any two points in terms of the Mach numbers at those locations. The energy equation, Eqn (2) says that the total temperature is constant in this flow, implying that:

$$\frac{T_2}{T_1} = \frac{T_2}{T_0} \frac{T_0}{T_1} \quad (67)$$

Using the definition of total temperature, the static temperature ratio becomes:

$$\boxed{\frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}} \quad (68)$$

From the continuity equation, Eqn (1), we may express a relationship involving pressure by using the definition of the speed of sound as follows:

$$\frac{\gamma P_1}{a_1^2} u_1 = \frac{\gamma P_2}{a_2^2} u_2 \quad (69)$$

The pressure ratio may then be defined as:

$$\frac{P_2}{P_1} = \frac{M_1}{M_2} \frac{a_2}{a_1} = \frac{M_1}{M_2} \sqrt{\frac{T_2}{T_1}} \quad (70)$$

Using Eqn (68), the static pressure ratio becomes:

$$\boxed{\frac{P_2}{P_1} = \frac{M_1}{M_2} \sqrt{\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}}} \quad (71)$$

The density ratio may be obtained from the ideal gas law and the known pressure and temperature ratios.

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1} \frac{T_1}{T_2} \quad (72)$$

The static density ratio becomes:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2} \sqrt{\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}}} \quad (73)$$

Total pressure ratio is known from the definition of total pressure and the static pressure ratio:

$$\frac{P_{0,2}}{P_{0,1}} = \frac{P_2}{P_1} \left(\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2} \right)^{\gamma/\gamma-1} \quad (74)$$

Substituting the static pressure ratio from Eqn (68) and carefully cleaning up the exponents of the Mach term yields:

$$\boxed{\frac{P_{0,2}}{P_{0,1}} = \frac{M_1}{M_2} \left(\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2} \right)^{(\gamma+1)/2(\gamma-1)}} \quad (75)$$

In this case, M_2 is typically found by using Eqn (66) to determine the choke length, L_1^* , associated with the inlet Mach number, M_1 , and then computing the length to the choke length from the outlet via Eqn (76):

$$\boxed{L_2^* = L_1^* - L} \quad (76)$$

The outlet Mach number, M_2 , can then be found inversely from Eqn (66) using the known L_2^* . Property ratios between any point in the duct and the choke point can also be found by using Eqns (68), (71), (73), and (75) and substituting in “*” for point 1:

$$\boxed{\frac{T}{T^*} = \frac{(\gamma + 1)}{2 + (\gamma - 1)M^2}} \quad (77)$$

$$\boxed{\frac{P}{P^*} = \frac{1}{M} \sqrt{\frac{(\gamma + 1)}{2 + (\gamma - 1)M^2}}} \quad (78)$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{1}{M} \sqrt{\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)}}} \quad (79)$$

$$\boxed{\frac{P_0}{P_0^*} = \frac{1}{M} \left(\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)} \right)^{(\gamma+1)/2(\gamma-1)}} \quad (80)$$